BDDC preconditioners for non-conforming polytopal hybrid discretisation methods. Part II: The preconditioner.

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Motivation

Polytopal methods

- Increasingly popular choice for dealing with **complex geometries**.
- Often **non-conforming** and **hybrid** in nature (HHO, HDG, non-conforming VEM, ...).
- Efficient solvers for the resulting **large sparse linear systems** are needed.

Balancing domain decomposition by constraints (BDDC)

- Popular family of **robust** and **scalable** preconditioners.
- Traditional robustness proofs rely on **continuous trace and lifting inquequalities**, which are not available for non-conforming methods.

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Hybrid spaces

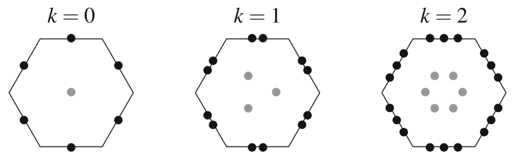
We consider the **hybrid space** $\underline{U}_h = U_h \times U_h^\partial$, where

$$U_h \doteq \bigtimes_{t\in\mathcal{T}_h} \mathbb{P}_k(t), \quad U_h^\partial \doteq \bigtimes_{f\in\mathcal{F}_h} \mathbb{P}_k(f), \quad U_h^{\partial,\mathrm{bd}} \doteq \bigtimes_{f\in\mathcal{F}_h^{\mathrm{bd}}} \mathbb{P}_k(f)$$

For each $\underline{v}_h = ((v_t)_{t\in\mathcal{T}_h}, (v_f)_{f\in\mathcal{F}_t}) \in \underline{U}_h$ we define the discrete $H^1(\Omega)$-seminorm

$$|\underline{v}_h|_{1,h} \doteq \left(\sum_{t\in\mathcal{T}_h} |\underline{v}_h|_{1,t}^2\right)^{1/2}$$

$$|\underline{v}_h|_{1,t}^2 \doteq \|\nabla v_t\|_{L^2(t)}^2 + \sum_{f\in\mathcal{F}_t} h_t^{-1}\|v_f - v_t\|_{L^2(f)}^2.$$



Discrete trace theory (I) [1]

We define the discrete $H^{1/2}(\partial\Omega)$-seminorm

$$|w_h|^2_{1/2,h} \doteq \sum_{f \in \mathcal{F}_h^{\mathrm{bd}}} h_f^{-1} \|w_f - \overline{w}_f\|^2_{L^2(f)} + \sum_{(f,f') \in \mathcal{F}\mathcal{F}_h^{\mathrm{bd}}} |f|_{d-1} |f'|_{d-1} \frac{|\overline{w}_f - \overline{w}_{f'}|^2}{|x_f - x_{f'}|^d},$$

Theorem: Trace inequality

$$|\gamma(\underline{v}_h)|_{1/2,h} \lesssim |\underline{v}_h|_{1,h} \qquad \forall \underline{v}_h \in \underline{U}_h$$

Theorem: Lifting inequality

There exists a lifting $\mathcal{L}_h : U_h^{\partial,\mathrm{bd}} \to \underline{U}_h$ such that

$$|\mathcal{L}_h(w_h)|_{1,h} \lesssim |w_h|_{1/2,h} \qquad \forall w_h \in U_h^{\partial,\mathrm{bd}}$$

[1] S. Badia, J. Droniou, J. Tushar, "A discrete trace theory for non-conforming polytopal hybrid discretisation methods.", submitted.

Discrete trace theory (II)

Theorem: Truncation estimate

Consider $\Gamma = \bigcup_{f \in \mathcal{F}_h(\Gamma)} \mathrm{cl}(f) \subset \partial\Omega$ and $\underline{v}_h \in \underline{U}_h$ such that

$$\int_\Gamma \gamma(\underline{v}_h) = 0$$

Define $w_h \in U_h^{\partial,\mathrm{bd}}$ as

$$\forall f \in \mathcal{F}_h^{\mathrm{bd}} \quad w_f = \begin{cases} v_f & \text{if } f \in \mathcal{F}_h(\Gamma), \\ 0 & \text{if } f \notin \mathcal{F}_h(\Gamma). \end{cases}$$

Then, under regularity assumptions for Γ, we have

$$|w_h|_{1/2,h} \lesssim \left(1 + \ln\left(\frac{\mathrm{diam}(\Omega)}{h}\right)\right) |\underline{v}_h|_{1,h},$$

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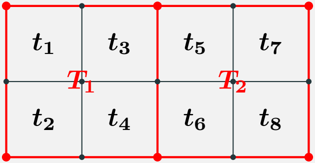
Domains and meshes

Fine mesh: For a domain Ω with boundary $\partial\Omega$, we consider

- A **partition** $\mathcal{T}_h$ of Ω into **polytopes** t.
- Its **skeleton** $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{bd}} \cup \mathcal{F}_h^{\mathrm{in}}$ with **faces** f.
- For each $t \in \mathcal{T}_h$, we denote by $\mathcal{F}_h(t)$ the subset of its faces.

Coarse mesh: we consider an agglomeration $(\mathcal{T}_H, \mathcal{F}_H)$ s.t.

- Each **subdomain** $T \in \mathcal{T}_H$ is a union of polytopes $t \in \mathcal{T}_h(T)$.
- Each **coarse face** $F \in \mathcal{F}_H$ is a union of faces $f \in \mathcal{F}_h(F)$.
- For each $T \in \mathcal{T}_H$, we denote by $\mathcal{F}_H(T)$ the subset of its macro-faces.



Sub-assembled spaces

We consider our fine mesh and it's coarse counterpart, $(\mathcal{T}_h, \mathcal{F}_h)$ and $(\mathcal{T}_H, \mathcal{F}_H)$ respectively.

- Original space - Continuous accross interfaces

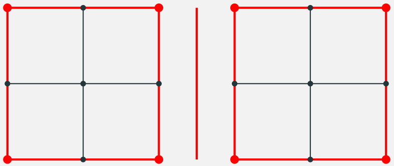
$$\underline{\widehat{U}}_h = U_h \times \widehat{U}_h^{\partial}$$

- Sub-assembled space - Discontinuous accross interfaces

$$\underline{U}_h = \bigtimes_{T \in \mathcal{T}_H} \underline{\widehat{U}}_h(T) \simeq U_h \times U_h^{\partial} \,, \quad U_h^{\partial} = \bigtimes_{T \in \mathcal{T}_H} \widehat{U}_h^{\partial}(T)$$

- Bubble spaces - Zero on coarse faces

$$\underline{U}_{h,0} = U_h \times U_{h,0}^{\partial}$$



Coarse constraints

We define for each coarse face $F \in \mathcal{F}_H$ a set of functionals

$$\Lambda_F = \left\{\lambda_F^i : U_h^\partial(F) \to \mathbb{R}\right\}_{i=1:n_c^F} \qquad \Lambda = \bigcup_{F \in \mathcal{F}_H} \Lambda_F$$

In our case, we will take **averages on coarse faces**, i.e

$$\lambda_F(\underline{v}_h) = \frac{1}{|F|} \int_F \gamma(\underline{v}_h)$$

We can then define the BDDC space as the space of functions that are discontinous accross interfaces, but with **continuous coarse degrees of freedom** (DoFs).

$$\underline{\widetilde{U}}_h = U_h \times \widetilde{U}_h^\partial$$

in particular, we have

$$\lambda_F(\underline{v}_h(T)) = \lambda_F(\underline{v}_h(T')) \quad \text{for } T, T' \text{ neighbors of } F.$$

Abstract problem formulation

We consider $\mathcal{A}$ a coercive differential operator on Ω, and l a linear functional. We assume homogeneous Dirichlet boundary conditions.

Continuous problem: Find $u \in U$ such that for all $v \in U_0$ we have

$$\langle \mathcal{A}u, v \rangle = \langle l, v \rangle$$

We define:

- $\mathcal{A}_h$ the discrete approximation of $\mathcal{A}$ on $\underline{U}_h$.
- $\widehat{\mathcal{A}}_h$ the restriction of $\mathcal{A}_h$ on $\widehat{\underline{U}}_h$.
- $\widetilde{\mathcal{A}}_h$ the restriction of $\mathcal{A}_h$ on $\widetilde{\underline{U}}_h$.
- $\mathcal{A}_{h,0}$ the restriction of $\mathcal{A}_h$ on $\underline{U}_{h,0}$.

Discrete problem: Find $\underline{u}_h \in \widehat{\underline{U}}_h$ such that for all $\underline{v}_h \in \widehat{U}_{h,0}$ we have

$$\langle \widehat{\mathcal{A}}_h \underline{u}_h, \underline{v}_h \rangle = \langle l_h, \underline{v}_h \rangle$$

Discrete harmonic spaces

Define the **discrete harmonic space** $\underline{H}_h$ as

$$\underline{H}_h \perp_{\mathcal{A}_h} \underline{U}_{h,0}$$

Define the **harmonic extension operator** $\mathcal{E}_h : \underline{U}_h \to \underline{H}_h$ such that given $\underline{w}_h \in \underline{U}_h$, we have

$$\langle \mathcal{A}_h \mathcal{E}_h \underline{w}_h, \underline{v}_h \rangle = 0 \quad \forall \underline{v}_h \in \underline{U}_{h,0}$$
$$\mathcal{E}_h \underline{w}_h = \underline{w}_h \text{ on } \mathcal{F}_H$$

and it's restriction $\widehat{\mathcal{E}_h}$ to $\underline{\widehat{U}}_h$.

- Functions in $\underline{H}_h$ are fully determined by their values on the interface.
- Interior DoFs are obtained by solving local problems with Dirichlet BCs given by the interface DoFs.
- $\mathcal{E}_h$ is the minimum energy lifting w.r.t $\mathcal{A}_h$.

Weighting operator

Define the **weighting operator** $W_h : \underline{U}_h \to \widehat{\underline{U}}_h$ such that

- $W_h I_h$ is the identity on U_h (interior DoFs)
- $W_h I_h$ is a projection on $\widehat{U}_h^\partial$ (interface DoFs)

where $I_h : \widehat{\underline{U}}_h \to \underline{U}_h$ is the natural injection.

In our case, we choose the weighted average

$$(W_h \underline{u}_h)|_F = \frac{1}{2}\left(\underline{u}_h(T)|_F + \underline{u}_h(T')|_F\right) \ \forall F \in \mathcal{F}_H$$

where T and T' are the two subdomains sharing the face F.

Preconditioner

We define the BDDC preconditioner $\widehat{\mathcal{B}}_h : \underline{\widehat{U}}_h^* \to \underline{\widehat{U}}_h$ as

$$\widehat{\mathcal{B}}_h = \mathcal{Q}_h \widetilde{\mathcal{A}}_h^{-1} \mathcal{Q}_h^T + I_0 \mathcal{A}_{h,0}^{-1} I_0^T$$

with $\mathcal{Q}_h = \widehat{\mathcal{E}_h} W_h : \underline{\widetilde{U}}_h \to \underline{\widehat{H}}_h$ and $I_0 : \underline{U}_{h,0} \to \underline{\widehat{U}}_h$ the natural injection.

Theorem: Spectral equivalence

Under the assumption that $||\cdot||_{\widetilde{\mathcal{A}}_h} \simeq ||\cdot||_{1,h}$

$$\kappa\left(\widehat{\mathcal{B}}_h \widehat{\mathcal{A}}_h\right) \lesssim \left(1 + \ln \frac{H}{h}\right)^2$$

where the hidden constant does not depend on H, h and the number of subdomains.

Proof: Spectral equivalence (I)

From abstract aditive Schwartz theory [1] , we have

$$\kappa\left(\widehat{\mathcal{B}}_h\widehat{\mathcal{A}}_h\right) \leqslant \max\left\{1, \sup_{\underline{v}_h\in\underline{\widetilde{U}}_h} \frac{\|\mathcal{Q}_h\underline{v}_h\|^2_{\widetilde{\mathcal{A}}_h}}{\|\underline{v}_h\|^2_{\widetilde{\mathcal{A}}_h}}\right\}$$

$$\leqslant \max\left\{1, 1 + \sup_{\underline{v}_h\in\underline{\widetilde{H}}_h} \frac{\|\mathcal{Q}_h\underline{v}_h - \underline{v}_h\|^2_{\widetilde{\mathcal{A}}_h}}{\|\underline{v}_h\|^2_{\widetilde{\mathcal{A}}_h}}\right\}.$$

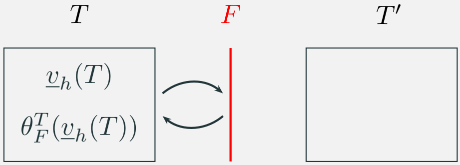
Let $\underline{w}_h = \mathcal{Q}_h\underline{v}_h - \underline{v}_h$.

[1] J. Mandel, B. Sousedík and C. R. Dohrmann, *Multispace and multilevel BDDC*, Springer (2008)

Proof: Spectral equivalence (II)

We define the face restriction operators θ_F^T which takes a function $\underline{v}_h$ and

1. Restrict a function to the macro face F.
2. Extends it by zero to ∂T
3. Extends it harmonically to T



Corollary: Truncation estimate

Let $F \in \mathcal{F}_H(T)$ and $\underline{v}_h \in \underline{U}_h$ which satisfies $\int_F \gamma(\underline{v}_h) = 0$. Then following estimate holds:

$$|\gamma(\theta_F^T(\underline{v}_h))|^2_{1/2,h} \lesssim (1 + \ln(H/h))^2 |\underline{v}_h|^2_{1,h}.$$

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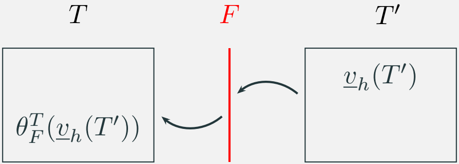

Corollary: Truncation estimate

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$$|\gamma(\theta_F^T(\underline{v}_h))|_{1/2,h}^2 \lesssim (1 + \ln(H/h))^2 |\underline{v}_h|_{1,h}^2.$$

Proof: Spectral equivalence (III)

One can see that for $\underline{w}_h = \mathcal{Q}_h \underline{v}_h - \underline{v}_h$ we have

$$\underline{w}_h(T) = \sum_{F \in \mathcal{F}_H(T)} \theta_F^T(\underline{w}_h) \,, \quad \theta_F^T(\underline{w}_h) = \frac{1}{2} \left[\theta_F^T(\underline{v}_h(T')) - \theta_F^T(\underline{v}_h(T)) \right]$$

Then using the above and triangle inequalities, we get

$$\| \mathcal{Q}_h \underline{v}_h - \underline{v}_h \|_{\widetilde{\mathcal{A}}_h}^2 \lesssim \sum_{T \in \mathcal{T}_H} \sum_{F \in \mathcal{F}_H(T)} \langle \widetilde{\mathcal{A}}_h \theta_F^T(\underline{w}_h), \theta_F^T(\underline{w}_h) \rangle,$$

Each of the terms can be bounded by

$$\langle \widetilde{\mathcal{A}}_h \theta_F^T(\underline{w}_h), \theta_F^T(\underline{w}_h) \rangle \lesssim \| \theta_F^T(\underline{v}_h(T) - \alpha_F) \|_{\widetilde{\mathcal{A}}_h, T}^2 + \| \theta_F^T(\underline{v}_h(T') - \alpha_F) \|_{\widetilde{\mathcal{A}}_h, T}^2$$

where we have added and substracted a constant $\alpha_F \in \mathbb{R}$.

Proof: Spectral equivalence (IV)

Since $\underline{v}_h \in \underline{\widetilde{H}}_h$, we can choose $\alpha_F = \lambda_F(\underline{v}_h(T)) = \lambda_F(\underline{v}_h(T'))$ so that

$$\int_F \gamma \left(\underline{v}_h(T') - \alpha_F\right) = 0$$

Then

$$\begin{aligned}
\|\theta_F^T(\underline{v}_h(T') - \alpha_F)\|^2_{\widetilde{\mathcal{A}}_h,T} &\lesssim \|\theta_F^T(\underline{v}_h(T') - \alpha_F)\|^2_{1,h} && \triangleright \text{ norm eq.} \\
&\lesssim |\underline{v}_h(T')|_F|^2_{1/2,h} && \triangleright \text{ lifting} \\
&\lesssim (1 + \ln(H/h))^2 \|\underline{v}_h(T')\|^2_{1,h} && \triangleright \text{ truncation} \\
&\lesssim (1 + \ln(H/h))^2 \|\underline{v}_h(T')\|^2_{\widetilde{\mathcal{A}}_h,T'} && \triangleright \text{ norm eq.}
\end{aligned}$$

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Problem setup

We solve the Poisson problem with Dirichlet bcs on $\Omega = [0,1]^2$

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

We use $N_c \times N_c$ simplexified cartesian meshes, and their Voronoi counterparts, partitioned into $N_p \times N_p$ processors.

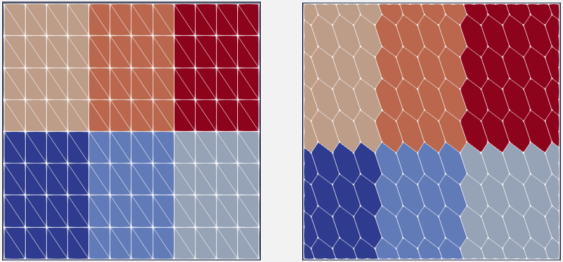

Figure 1: Left: simplexified mesh. **Right:** Voronoi mesh. Distributed amongst $N_p = 6$ processors, with colors mapping to the processor id.

Problem setup

We test our preconditioner for several popular schemes in the literature:

- Hybrid High-Order (HHO) [1]
- Mixed-order HHO [2]
- Hybridazable Discontinuous Galerkin (HDG) [3,4]

[1] D. A. Di Pietro and A. Ern, *A hybrid high-order locking-free method for linear elasticity on general meshes*, CMAME (2015)

[2] M. Cicuttin, A. Ern and N. Pigneo, *Hybrid High-Order Methods: A Primer with Applications to Solid Mechanics*, Springer (2021)

[3] B. Cockburn, J. Gopalakrishnan and R. Lazarov, *Unified Hybridization of Discontinuous Galerkin, Mixed, and Continuous Galerkin Methods for Second Order Elliptic Problems*, SIAM-NA (2009)

[4] S. Du and F. Sayas, *A note on devising HDG+ projections on polyhedral elements*, arXiv (2020)

Benchmarks

Implementation Julia, using the **Gridap**[1] ecosystem. MPI support through **GridapDistributed.jl** and **PartitionedArrays.jl** [2]. Solvers implemented in **GridapSolvers.jl**[3].

We perform **weak scalability** tests, that is

- constant local problem size, $N_c/N_p \sim H/h$
- increasing N_p up to 768 processors

All experiments are run on the Gadi supercomputer at NCI.

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[1] S. Badia and F. Verdugo, *Gridap: An extensible Finite Element toolbox in Julia*, JOSS (2020)

[2] S. Badia and A. F. Martín and F. Verdugo, *GridapDistributed: a massively parallel finite element toolbox in Julia*, JOSS (2022)

[3] J. Manyer and A. F. Martín and S. Badia, *GridapSolvers.jl: Scalable multiphysics finite element solvers in Julia*, JOSS (2024)

Results (I)

HDG of structured meshes

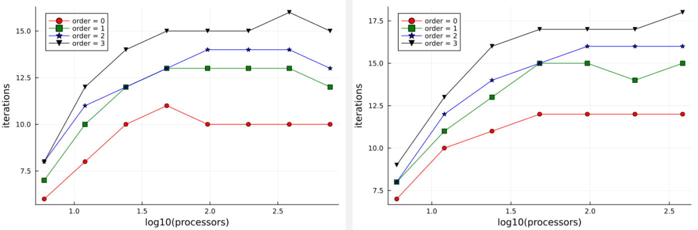


Figure 2: Number of CG iterations, as a function of the number of processors. Cases: HDG with $H/h = 8$ (left) and $H/h = 16$ (right).

Results (II)

HHO of structured meshes

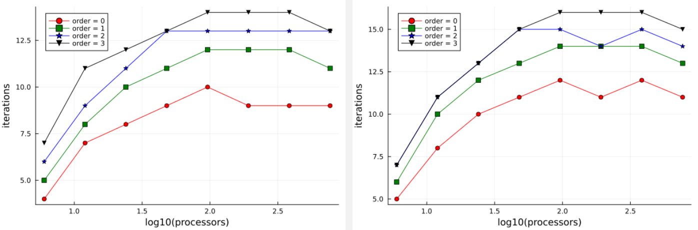


Figure 3: Number of CG iterations, as a function of the number of processors. Cases: HHO with H/h = 8 (left) and H/h = 16 (right).

Results (III)

Mixed-order HHO of structured meshes

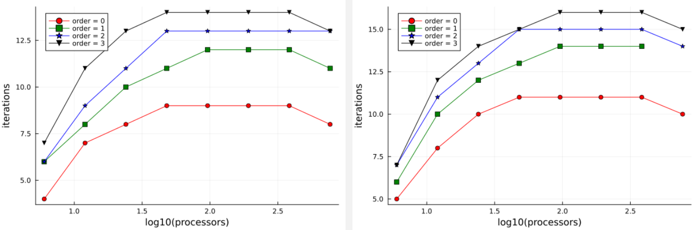
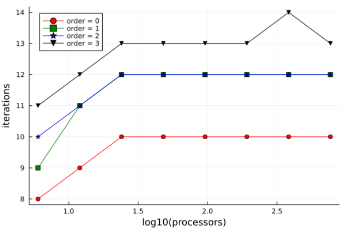


Figure 4: Number of CG iterations, as a function of the number of processors. Cases: Mixed-order HHO with $H/h = 8$ (left) and $H/h = 16$ (right).

Results (IV)

HDG+ of voronoi meshes



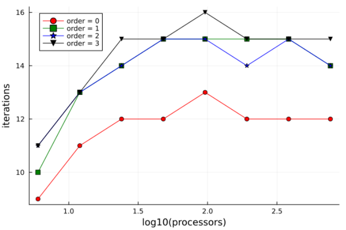


Figure 5: Number of CG iterations, as a function of the number of processors. Cases: HDG+ with $H/h = 8$ (left) and $H/h = 16$ (right).

Results (V)

HHO of voronoi meshes

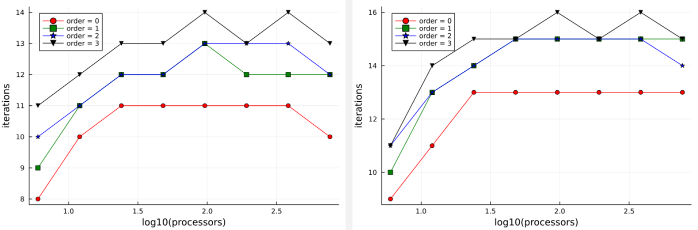


Figure 6: Number of CG iterations, as a function of the number of processors. Cases: HHO with $H/h = 8$ (left) and $H/h = 16$ (right).

Results (VI)

Mixed-order HHO of voronoi meshes

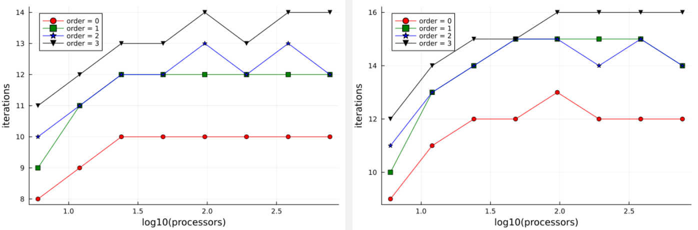


Figure 7: Number of CG iterations, as a function of the number of processors. Cases: Mixed-order HHO with H/h = 8 (left) and H/h = 16 (right).

Thank you!

S. Badia, J. Droniou, J. Manyer, J. Tushar, “Balancing domain decomposition by constraints preconditioners for non-conforming hybrid discretisation methods”, in preparation.

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J. Manyer, A. Martin, S. Badia “GridapSolvers: Scalable multiphysics finite element solvers in Julia“, Journal of Open Source Software, 2024.

NCI AUSTRALIA

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[2] Santiago Badia, Jérôme Droniou, and Jai Tushar. ***A discrete trace theory for non-conforming polytopal hybrid discretisation methods.*** 2024. arXiv: 2409.15863 [math.NA].

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[4] Santiago Badia, Alberto F. Martín, and Francesc Verdugo. **"GridapDistributed: a massively parallel finite element toolbox in Julia".** In: *Journal of Open Source Software* 7.74 (2022), p. 4157. DOI: 10.21105/joss.04157.

[5] Santiago Badia and Francesc Verdugo. **"Gridap: An extensible Finite Element toolbox in Julia".** In: *Journal of Open Source Software* 5.52 (2020), p. 2520. DOI: 10.21105/joss.02520.

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[13] Daniele Antonio Di Pietro and Jérôme Droniou. ***The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications*****.** Vol. 19. MS&A. Springer, 2020. DOI: 10.1007/978-3-030-37203-3.

Appendix A - Implementation details for BDDC (I)

Our preconditioner is given by

$$\widehat{\mathcal{B}}_h = \mathcal{Q}_h \widetilde{\mathcal{A}}_h^{-1} \mathcal{Q}_h^T + I_0 \mathcal{A}_{h,0}^{-1} I_0^T$$

Given a residual $\widehat{r}_h$, it returns a correction $\widehat{z}_h = \widetilde{z}_h + z_{h,0}$ given by

$$\widetilde{z}_h = \mathcal{Q}_h \widetilde{\mathcal{A}}_h^{-1} \mathcal{Q}_h^T \widehat{r}_h \ , \quad z_{h,0} = I_0 \mathcal{A}_{h,0}^{-1} I_0^T \widehat{r}_h$$

We can further split the problem by considering $\underline{\widetilde{U}}_h = \widetilde{U}_{h,0} \bigoplus \widetilde{U}_{h,c}$ where

- $\widetilde{U}_{h,0}$ is the subset of $\underline{\widetilde{U}}_h$ with zero coarse DoF values.
- $\widetilde{U}_{h,c}$ is the complementary, i.e the subset of $\underline{\widetilde{U}}_h$ with at least one non-zero coarse DoF value.

We can write $\widetilde{z}_h = \mathcal{Q}_h(\widetilde{z}_{h,0} + \widetilde{z}_{h,c})$ with

$$\widetilde{z}_{h,0} = \widetilde{\mathcal{A}}_{h,0}^{-1} r_h \ , \quad \widetilde{z}_{h,c} = \widetilde{\mathcal{A}}_{h,c}^{-1} r_h \ , \quad r_h = \mathcal{Q}_h^T \widehat{r}_h$$

Appendix A - Implementation details for BDDC (II)

Let A be the matrix representation of $\mathcal{A}_h$, i.e the sub-assembled system matrix. We have that

- A is block-diagonal, with blocks A^k corresponding to subdomains.
- Each block A^k can be locally split into interior/interface blocks, i.e

$$A^k = \begin{bmatrix} A^k_{II} & A^k_{I\Gamma} \\ A^k_{\Gamma I} & A^k_{\Gamma\Gamma} \end{bmatrix}$$

Also:

- Let C^k be the matrix representation of the coarse constraints, local to each processor.
- Let $R_I = \text{diag}(R^k_I)$ and $R_\Gamma = \text{diag}(R^k_\Gamma)$ be the interior and interface restrictions, respectively.
- Let W be the matrix representation of W_h.

Appendix A - Implementation details for BDDC (III)

Interior correction

- The block-diagonal matrix $A_{II} = \text{diag}(A_{II}^k)$ is the matrix representation of $\mathcal{A}_{h,0}$.
- The block-diagonal matrix $R_I = \text{diag}(R_I^k)$ is the matrix representation of I_0^T.

Then if we take $z_0, \widehat{r}$ the vector representations of $z_{h,0}, \widehat{r}_h$ respectively, we have

$$z_0 = R_I^T A_{II}^{-1} R_I \widehat{r}$$

which can be solved locally on each processor.

Appendix A - Implementation details for BDDC (IV)

Fine correction

The fine correction

$$\widetilde{z}_{h,0} = \widetilde{\mathcal{A}}_{h,0}^{-1} r_h$$

can be computed locally by solving

$$\begin{bmatrix} A_{II}^k & A_{I\Gamma}^k & \\ A_{\Gamma I}^k & A_{\Gamma\Gamma}^k & (C^k)^T \\ & C^k & \end{bmatrix} \cdot \begin{bmatrix} \widetilde{z}_{0,I}^k \\ \widetilde{z}_{0,\Gamma}^k \\ \lambda^k \end{bmatrix} = \begin{bmatrix} r_I^k \\ r_\Gamma^k \\ 0 \end{bmatrix}$$

where $\widetilde{z}_0, r$ the vector representations of $\widetilde{z}_{h,0}, r_h$ respectively, and

$$\begin{bmatrix} \widetilde{z}_{0,I}^k \\ \widetilde{z}_{0,\Gamma}^k \end{bmatrix} = \begin{bmatrix} R_I^k \\ R_\Gamma^k \end{bmatrix} \cdot \widetilde{z}_0^k \,, \quad \begin{bmatrix} r_I^k \\ r_\Gamma^k \end{bmatrix} = \begin{bmatrix} R_I^k \\ R_\Gamma^k \end{bmatrix} \cdot r^k$$

Appendix A - Implementation details for BDDC (V)

Coarse correction

We have $\widetilde{U}_{h,c} = \text{span}\{\phi_\lambda\}_{\lambda\in\Lambda}$, where ϕ_λ is the global basis function that is one on λ and zero on all other coarse degrees of freedom. Denote by Φ the matrix that has columns ϕ_λ.

We can compute Φ^k, by solving the local problems

$$\begin{bmatrix} A^k_{II} & A^k_{I\Gamma} & \\ A^k_{\Gamma I} & A^k_{\Gamma\Gamma} & (C^k)^T \\ & C^k & \end{bmatrix} \cdot \begin{bmatrix} \Phi^k_I \\ \Phi^k_\Gamma \\ \lambda^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_d \end{bmatrix}$$

The matrix representation of $\widetilde{\mathcal{A}}_{h,c}$ is then given by

$$\widetilde{A}_c^{-1} = \Phi S_c^{-1} \Phi^T \ , \quad S_c = \sum_k S_c^k = \sum_k (\Phi^k)^T A^k \Phi^k$$

Appendix A - Implementation details for BDDC (VI)

Coarse correction[1]

Generally, S_c is assembled into a single processor and factorized. The coarse correction then involves:

1. gathering the local coarse residuals $r_c^k = (\Phi^k)^T r^k$ into a single processor,
2. solving the coarse problem, and
3. scattering the result back to each processor and doing a linear combination with the basis, to obtain $\tilde{z}_c^k$.

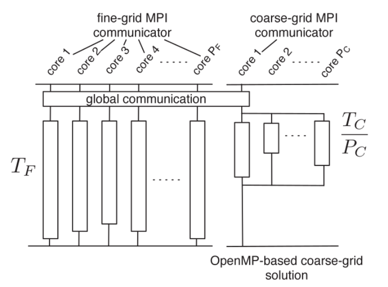


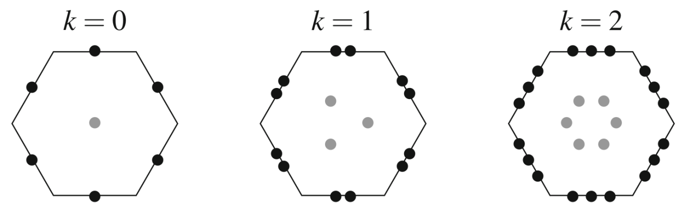
Figure 8: Processor layout for computing BDDC corrections.

[1] S. Badia and A. F. Martín and J. Principe, *A Highly Scalable Parallel Implementation of Balancing Domain Decomposition by Constraints*, SIAM J. Sci. Comput. (2014).

Appendix B - HHO formulation for the Poisson problem (I) [1] [2]

Given a polynomial order k, we consider the **local** and **global** HHO spaces

$$\underline{U}_T^k = \left\{ (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathbb{P}^k(T),\ v_F \in \mathbb{P}^k(F)\ \forall F \in \mathcal{F}_T \right\}$$
$$\underline{U}_h = \left\{ ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : v_T \in \mathbb{P}^k(T)\ \forall T \in \mathcal{T}_h,\ v_F \in \mathbb{P}^k(F)\ \forall F \in \mathcal{F}_h \right\}$$



[1] D. Di Pietro and J. Droniou, *The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications*, Springer, 2020.

[2] M. Cicuttin, A. Ern and N. Pignet, *Hybrid High-Order Methods: A Primer with Applications to Solid Mechanics*, Springer, 2021.

Appendix B - HHO formulation for the Poisson problem (II)

Potential reconstruction operator

We define $R_T^{k+1}\underline{v}_T : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$ such that $\forall w \in \mathbb{P}^{k+1}(T)$ we have

$$\left(\nabla R_T^{k+1}\underline{v}_T, \nabla w\right)_T = (\nabla v_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \nabla w \cdot \boldsymbol{n}_{TF})_F$$

$$\left(R_T^{k+1}\underline{v}_T, w\right)_T = (v_T, w)_T$$

Difference operators

We define $\delta_T^k \underline{v}_T : \underline{U}_T^k \to \mathbb{P}^k(T)$ and $\delta_F^k \underline{v}_F : \underline{U}_T^k \to \mathbb{P}^k(F)$ as

$$\delta_T^k \underline{v}_T = \pi_T^k (R_T^{k+1}\underline{v}_T - v_T)\ , \quad \delta_{TF}^k \underline{v}_T = \pi_F^k (R_T^{k+1}\underline{v}_T - v_F)$$

Appendix B - HHO formulation for the Poisson problem (III)

The global problem is then given by: Find $\underline{u}_h \in \underline{U}_{h,0}$ such that $\forall \underline{v}_h \in \underline{U}_{h,0}$

$$a_h(\underline{u}_T, \underline{v}_T) = b_h(\underline{v}_T)$$

with

$$a_h(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) \ , \quad b_h(\underline{v}_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T \ ,$$

and where the local contributions are given by

$$\begin{aligned} a_T(\underline{u}_T, \underline{v}_T) &= \left(\nabla R_T^{k+1} \underline{u}_T, \nabla R_T^{k+1} \underline{v}_T\right)_T + s_T(\underline{u}_T, \underline{v}_T) \\ s_T(\underline{u}_T, \underline{v}_T) &= \sum_{F \in \mathcal{F}_T} h_F^{-1} \left[\left(\delta_{TF}^k - \delta_T^k\right) \underline{u}_T, \left(\delta_{TF}^k - \delta_T^k\right) \underline{v}_T\right] \end{aligned}$$